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Correlation functions near the critical point

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Abstract. Using renormalization group arguments we expand n-point correlation functions (for non-exceptional wavevectors) in expectation values of translational invariant short-range operators O_i. We use the fact that the Fourier components of our operators become negligible for wavevectors q large in comparison to the momentum cut-off.

The correlation functions show the same non-analyticities at the critical point as the expectation values $\langle O_i \rangle$. The expansion coefficients are regular in the thermodynamic variables for $q \neq 0$. They can be expressed in terms of (a) functions which become singular at $q = 0$ and yield the scaling behaviour, and (b) functions which are regular at $q = 0$. The expansion coefficients of the two-point correlation function are sums of both types of functions.

1. Introduction

The static correlation functions show two characteristic features near the critical point: (a) they obey scaling and (b) they show a non-analytic behaviour even for finite wavelengths as a function of the thermodynamic variables. These properties are discussed on the basis of renormalization group (RG) ideas (Wilson 1971, Wilson and Kogut 1974) in this paper.

Consider the free energy F of the Hamiltonian

$$H = H\{g\} + \sum_j \kappa_j(q_j) O_j(q_j) \tag{1.1}$$

(a factor $-1/k_B T$ is incorporated in H and F), where $H\{g\}$ is translational invariant and parametrized by scaling fields g (Wegner 1972). The terms $\kappa(q)O(q)$ are perturbations of wavelength q. We find that the free energy of this system equals the free energy $F\{\tilde{g}\}$ of the translational invariant Hamiltonian $H\{\tilde{g}\}$ where we have an expansion

$$\tilde{g}_i = g_i + \tfrac{1}{2} \sum M_{ijk}(q) \kappa_j(q) \kappa_k(-q) + O(\kappa^3). \tag{1.2}$$

The coefficients M can be calculated from the RG equation. This confirms and refines a conjecture by Fisher (1962) (compare Riedel and Wegner 1969) according to which inhomogeneous perturbations to a system near criticality leave the non-analytic structure of the free energy basically unchanged but change the critical parameters, such as critical temperature. Indeed the condition for criticality is $\tilde{g}_i = 0$ for all relevant operators. Therefore at criticality we have $g_i = -\frac{1}{2}\Sigma M\kappa\kappa + \dots$ (actually Fisher considered an Ising antiferromagnet in a homogeneous magnetic field, but this is equivalent to an Ising ferromagnet in a staggered magnetic field). Differentiating F with respect to κ_j

and κ_k one obtains the representation for the correlation function ($q \neq 0$):

$$\langle O_j(q) O_k(-q) \rangle = \sum_i M_{ijk}(q) \langle O_i \rangle. \tag{1.3}$$

Therefore the correlation function (1.3) shows the same non-analyticities as a function of the thermodynamic variables as the expectation values $\langle O_i \rangle = \langle \delta H / \partial g_i \rangle$.

The expansion coefficients M consist of two contributions

$$M_{ijk} = R_{ijk} + S_{ijk} \tag{1.4}$$

where R is a regular function of q and of the g's (provided no logarithmic singularities appear; logarithmic corrections will not be discussed in this paper), whereas the singular part S obeys scaling:

$$S_{ijk}(\boldsymbol{q}, \{g_r\}) = q^{y_i - y_j - y_k} S_{ijk}(\boldsymbol{e}, \{g_r / q^{y_r}\}) \tag{1.5}$$

with $\boldsymbol{e} = \boldsymbol{q}/|q|$ (y's are the scaling exponents of the operators). S can be expanded in powers of g_r/q^{y_r}. If $y_j + y_k > d$ (d is the dimensionality of the system), then the leading scaling behaviour of the two-point correlation function near the critical point is given by

$$\sum_i S_{ijk}(q, \{g_r^{\text{rel}}\}) \langle O_i \rangle, \tag{1.6}$$

where i runs over the indices of *all* operators and g_r^{rel} runs only over the fields of the relevant operators (the fields of the irrelevant operators provide corrections to the scaling behaviour). This is in agreement with the expression for the spin–spin correlation function proposed by Fisher and Langer (1968),

$$\langle S_q S_{-q} \rangle = q^{-2+\eta}(A + B\tau/q^{1/\nu} + C|\tau|^{1-\alpha}/q^{(1-\alpha)/\nu} + \ldots) \tag{1.7}$$

($\tau = T - T_c$), where the first two terms come from $\langle 1 \rangle$ and the third term from the expectation value of the energy $\langle H - E_{\text{crit}} \rangle$. Fisher and Aharony (1973) showed that this *ansatz* is consistent with an expansion of the correlation function around dimensionality four and they determined the coefficients. Brezin *et al* (1974a) and Brezin *et al* (1974b) derived equation (1.7) from the Callan–Symanzik equation (compare Symanzik 1971) and generalized it to the case of a finite magnetic field and allowed for temperatures below T_c. They found a further contribution to the spin correlation function of the n-vector model ($n > 1$) which scales like $\langle S_i S_j - \delta_{ij} S^2/n \rangle$. This term differs from zero below T_c or for a finite magnetic field. Hecht (1967) calculated the spin–energy correlation in the two-dimensional Ising model and found $\langle S \rangle q^{-1}$. All of these contributions are contained in equation (1.6) which gives the general behaviour of the scaling part of the correlation function.

To obtain equation (1.2) we transform the Hamiltonian (1.1) using the RG procedure. This procedure has the following effects.

(i) It reduces the length scale by a factor e^l and therefore transforms the perturbations $O_j(q)$ into perturbations $e^{y_j l} O_j(qe^l)$. Within linear approximation this yields the scaling law for the correlation function if we bear in mind that the scaling fields g_r transform into $g_r e^{y_r l}$.

(ii) If one chooses a RG equation with smooth momentum cut-off of order q_0, then the perturbation $O_j(qe^l)$ becomes negligible for $qe^l \gg q_0$. Obviously in this limit the linear approximation breaks down. The nonlinear terms of the RG equation will generate perturbations $O_i((q+q')e^l)$ from the perturbations $O_j(qe^l)$ and $O_k(q'e^l)$. Again if $q + q' \neq 0$ then these perturbations are negligible for sufficiently large l.

(iii) If however $q+q' = 0$ then the RG procedure generates homogeneous perturbations. Since for large l all other perturbations become negligible, we may forget the perturbations with $q \neq 0$ for large l. Then we apply the inverse RG procedure to this translational invariant Hamiltonian until we return to $l = 0$ and obtain a Hamiltonian $H\{\tilde{g}\}$ with the expansion (1.2) for $\tilde{g}$. Since the free energy is conserved under the total of this transformation, we have $F = F\{\tilde{g}\}$.

In § 2 we introduce the RG equation with smooth momentum cut-off and the eigenoperators $O_i^*(q)$ of its linearized version. We derive the RG equation for the corresponding fields μ and λ (sources in field theoretic language). The fields μ describe the homogeneous Hamiltonian and the inhomogeneous perturbations are added with coupling constants λ:

$$H = H^* + \sum \mu_i O_i^* + \sum \lambda_i O_i^*(q_i). \tag{1.8}$$

To facilitate further derivations we eliminate the contributions nonlinear in μ by introduction of the scaling fields g in § 3. We transform the Hamiltonian to the form (1.1) where the perturbation $\Sigma\kappa_j O_j(q_j)$ transforms in linear order in κ but all order in μ resp. g into $\Sigma\kappa_j \mathrm{e}^{y_j l} O_j(q_j \mathrm{e}^{y_j l})$. This is a first step to introduce scaling field $f(q)$ for the inhomogeneous perturbations. In § 4 we derive the RG equation for κ. This enables us to obtain the equations for the coefficients M and a RG equation for the correlation function G in § 5. In § 6 we discuss the two-point correlation functions. Finally in § 7 we introduce the scaling fields $f(q)$ to higher order in κ and discuss the three-point correlations. We find that the regular part R of M is absorbed into f. This is in agreement with Ma's discussion (1974) of correlation functions in terms of scaling fields f. The singular part S of M appears in the expansion of $\tilde{g}$ in powers of f.

2. Renormalization group equations

In this section we formulate the RG equation given by Wilson and Kogut (1974) in a form suitable for our problem. This yields equations (2.17)–(2.21). Wilson derived (apart from some constants which he could neglect) the RG equation with smooth momentum cut-off:

$$\frac{\mathrm{d}H}{\mathrm{d}l} = dV\frac{\partial H}{\partial V} + \int \left(\frac{d}{2}S_q + q\nabla S_q\right)\frac{\delta H}{\delta S_q}\,\mathrm{d}^d q$$
$$+ \int \rho(q)\left(S_q\frac{\delta H}{\delta S_q} + \frac{\delta^2 H}{\delta S_q \delta S_{-q}} - \frac{\delta H}{\delta S_q}\frac{\delta H}{\delta S_{-q}} - 1\right)\mathrm{d}^d q \tag{2.1}$$

where S_q are the Fourier components of the classical variable $S(r)$, $S_q = \int \mathrm{d}^d r S(r)\,\mathrm{e}^{-\mathrm{i}qr}$. $\rho(q)$ is an appropriate function of q whose Fourier transform is of short range. The first term and the first integral on the right-hand side describe the change of the Hamiltonian under a simple change of the length scale $q \to q\,\mathrm{e}^l$. The second integral transforms the variables according to (Wegner 1974):

$$S_q \to S_q + \Delta l \rho(q)(S_q - \delta H/\delta S_{-q}). \tag{2.2}$$

If one chooses the representation

$$H = Vv_0 + v_1 S_0 + \tfrac{1}{2}\int v_2(q) S_q S_{-q}\,\mathrm{d}^d q + \frac{1}{3!}\int v_3(q_1, q_2) S_{q_1} S_{q_2} S_{-q_1-q_2}\,\mathrm{d}^d q_1\,\mathrm{d}^d q_2 + \dots \tag{2.3}$$

for the translational invariant system then the volume V enters only via the constant v_0. The transformation (2.1) is constructed in such a way that the free energy of the total system is conserved. The fixed-point Hamiltonian obeys

$$dH^*/dl = 0. \tag{2.4}$$

We add a perturbation $\mu_i O_i^*$ to H^* where O_i^* is translational invariant. This yields in linear order in μ_i a contribution $\mu_i L O_i^*$ to dH/dl with

$$L = \int d^d q(dS_q/2 + \rho(q)S_q + q\nabla S_q - 2\rho(q)\delta H^*/\delta S_{-q} + \rho(q)\delta/\delta S_{-q})\delta/\delta S_q. \tag{2.5}$$

We define the eigenperturbations O_i^* by

$$LO_i^* = y_i O_i^*. \tag{2.6}$$

Next we consider a local perturbation $\tilde{O}_i$ (an operator which decays within a distance $1/q_0$ from the origin) which obeys

$$L\tilde{O}_i = -x_i \tilde{O}_i. \tag{2.7}$$

The operator can be written as a functional of the components S_q. We construct the operator $O_i(r)$ by replacing any S_q in O_i by $S_q\,e^{iqr}$. One finds from equations (2.5) and (2.7) that

$$L\tilde{O}_i(r) = -x_i\tilde{O}_i(r) - r\nabla\tilde{O}_i(r). \tag{2.8}$$

Therefore a perturbation $\tilde{O}_i(r)$ transforms under the change of the length scale by e^l according to

$$\tilde{O}_i(r) \to e^{-x_i l}\tilde{O}_i(r\,e^{-l}). \tag{2.9}$$

From this equation we deduce that

$$O_i^*(q) = \int d^d r \tilde{O}_i(r)\,e^{-iqr} \tag{2.10}$$

transforms according to

$$O_i^*(q) \to e^{(d-x_i)l} O_i^*(q\,e^l) \tag{2.11}$$

and comparison with (2.6) for $q = 0$ yields ($O_i^* \equiv O_i^*(0)$):

$$y_i = d - x_i \tag{2.12}$$

provided that $O_i^*(0)$ does not vanish. Note that from equation (2.8) one derives

$$L\nabla\tilde{O}_i = -(x_i + 1)\nabla\tilde{O}_i. \tag{2.13}$$

In the following we will restrict ourselves to operators $O_i^*(q)$ with $O_i^*(0) \neq 0$ since the Fourier transform of $\nabla O_i^*(r)$ can be expressed by $O_i^*(q)$:

$$\int d^d r \nabla\tilde{O}_i(r)\,e^{-iqr} = iqO_i^*(q). \tag{2.14}$$

We return to equation (2.1) which is bilinear. Therefore a perturbation $\lambda_j O_j^*(q_j) + \lambda_k O_k^*(q_k)$ will add a contribution

$$\lambda_j\lambda_k \sum_i a_{ijk} O_i^*(q_j + q_k) \tag{2.15}$$

to $\mathrm{d}H/\mathrm{d}l$ with

$$-2\int \rho(q)\frac{\delta O_j^*(q_j)}{\delta S_q}\frac{\delta O_k^*(q_k)}{\delta S_{-q}}\,\mathrm{d}^d q$$
$$= \sum_k a'_{ijk}(q_j, q_k)O_i^*(q_j+q_k) + V\delta_{q_j+q_k,0}a'_{0jk}(q_j, q_k). \tag{2.16}$$

Now we are able to write down the equations for $\mathrm{d}\mu/\mathrm{d}l$ and $\mathrm{d}\lambda/\mathrm{d}l$.

The Hamiltonian

$$H_0 = H^* + \sum \mu_i O_i^* + \sum \lambda_i(q_i)O_i^*(q_i) \tag{2.17}$$

transforms into

$$H_l = H^* + \sum \mu_i(l)O_i^* + \sum \lambda_i(q_i\,\mathrm{e}^l, l)O_i^*(q_i\,\mathrm{e}^l) \tag{2.18}$$

with

$$\mathrm{d}\lambda_i(q)/\mathrm{d}l = y_i\lambda_i(q) + \tfrac{1}{2}\sum a'_{ijk}(q_1, q_2)\lambda_j(q_1)\lambda_k(q_2)\delta_{q,q_1+q_2} + \sum a'_{ijk}(q, 0)\lambda_j(q)\mu_k \tag{2.19}$$

$$\mathrm{d}\mu_i/\mathrm{d}l = y_i\mu_i + \tfrac{1}{2}\sum a'_{ijk}(0, 0)\mu_j\mu_k, \tag{2.20}$$

where

$$\mathrm{d}/\mathrm{d}l = \partial/\partial l + q\partial/\partial q. \tag{2.21}$$

The separation between $\lambda_i(0)$ and μ_i is arbitrary. We choose the separation so that $\mu_i(l)$ depends only on the initial values of $\mu_j(0)$. A constant perturbation to the Hamiltonian $V\mu_0$ is distinguished in so far as it enters on the right-hand side of equation (2.1) only in the first term which yields $y_0 = d$. Secondly only the Fourier component $q = 0$ contributes. Therefore equations (2.19) and (2.20) apply for μ_0 and $\lambda_0(0)$.

3. Scaling fields for the perturbations in linear order

If we neglect those terms in equation (2.19) which are quadratic in λ then we obtain

$$\mathrm{d}\lambda_i(q)/\mathrm{d}l = y_i\lambda_i(q) + \sum a'_{ijk}(q, 0)\lambda_i(q)\mu_k. \tag{3.1}$$

For $\mu_i \equiv 0$ we obtain the solution

$$\lambda_i(q, l) = \mathrm{e}^{y_i l}\lambda_i(q\,\mathrm{e}^{-l}, 0) \tag{3.2}$$

with (2.21). To take into account the second term on the right-hand side of equation (3.1) we introduce scaling fields $f_i(q)$ in analogy to the scaling fields g_i in Wegner (1972) and Wegner and Riedel (1973). In these references we expressed the fields μ_i in terms of g_i's:

$$\mu_i = \mu_i\{g\} = g_i + \tfrac{1}{2}\sum b_{ijk}g_j g_k + \dots \tag{3.3}$$

which obey exactly

$$\mathrm{d}g_i/\mathrm{d}l = y_i g_i. \tag{3.4}$$

Similarly we expand

$$f_i(q) = \sum_j p_{ij}(q)\lambda_j(q) + \mathrm{O}(\lambda^2) \tag{3.5}$$

and require

$$\mathrm{d}f_i(q)/\mathrm{d}l = \partial f_i/\partial l + q\nabla_q f_i = y_i f_i(q). \tag{3.6}$$

The terms of order λ^2 in equation (3.5) are necessary to take care of the terms of order λ^2 in equation (2.19). We will return to these terms in § 7. Here it is sufficient to consider the term linear in λ.

The expansion coefficients $p_{ij}(q)$ depend on the fields g. Therefore we obtain

$$\sum_j (D+q\nabla_q)p_{ij}(q)\lambda_j(q) + \sum_j p_{ij}y_j\lambda_j(q) + \sum_{jks} p_{ik}a'_{kjs}(q,0)\mu_s\lambda_j(q) = y_i \sum_j p_{ij}\lambda_j(q) \tag{3.7}$$

with

$$D = \sum_k y_k g_k \frac{\partial}{\partial g_k}. \tag{3.8}$$

Equating the coefficients of $\lambda_j(q)$ we find

$$(D+q\nabla_q+y_j-y_i)p_{ij}(q) + \sum_{ks} p_{ik}(q)a'_{kjs}(q,0)\mu_s\{g\} = 0. \tag{3.9}$$

Let us expand $p_{ij}(q)$ in powers of the scaling fields g:

$$p_{ij}(q) = c_{ij}(q) + \sum c_{ijk}(q)g_k + \mathrm{O}(g^2); \tag{3.10}$$

then we obtain

$$(q\nabla_q+y_j-y_i)c_{ij}(q) = 0 \tag{3.11}$$

$$(q\nabla_q+y_j-y_i+y_k)c_{ijk}(q) = -\sum_s c_{is}(q)a'_{sjk}(q,0), \tag{3.12}$$

etc. The solutions depend on the boundary conditions. We require that $f = \lambda$ for $g = 0$, that is

$$c_{ij}(q) = \delta_{ij} \tag{3.13}$$

which yields

$$(q\nabla_q+y_j-y_i+y_k)c_{ijk}(q) = -a'_{ijk}(q,0). \tag{3.14}$$

Equation (3.14) can be written in the form

$$(q\partial/\partial q+z)U(q) = I(q) \tag{3.15}$$

with $c = U$ and $I = -a'$. Since we will deal several times with equations of this type we give a discussion. The formal solution of this equation reads

$$U(q) = q^{-z}\int^q \mathrm{d}p\, p^{z-1}I(p). \tag{3.16}$$

The lower bound of this integral depends on the boundary condition. If we choose

$$\lim_{q\to\infty} U(q) = 0 \tag{3.17}$$

then we obtain

$$U_\infty(q) = -q^{-z}\int_q^\infty \mathrm{d}p\, p^{z-1}I(p) \tag{3.18}$$

provided that $I(p)$ decays sufficiently rapidly. On the other hand we may require (we will do so for equation (3.14)) that $I(p)$ behaves regularly at $q = 0$. Then we will split off all powers p^m from $I(p)$ with $m < -z$:

$$I(p) = \sum_m c_m p^m + I_R(p) \tag{3.19}$$

with

$$\lim_{p \to 0} (I_R(p)p^z) = 0. \tag{3.20}$$

This can be done provided that $I(p)$ can be expanded in a Taylor expansion and no power p^{-z} occurs. (If $I(p)$ contains a term proportional to p^{-z} then a logarithmic contribution $q^{-z} \ln q$ in $U(q)$ is inevitable; however we will not discuss these logarithmic terms here.) From equation (3.19) we obtain the solution regular at $q = 0$:

$$U_0(q) = -\sum c_m q^m/(m+z) + q^{-z} \int_0^q \mathrm{d}p\, p^{z-1} I_R(p). \tag{3.21}$$

The solutions $U_0(q)$ and $U_\infty(q)$ differ by the solution sq^{-z} of the homogeneous equation

$$(q\partial/\partial q + z)U_\mathrm{h}(q) = 0. \tag{3.22}$$

We determine the constant s by

$$\begin{aligned} U_\infty(q) &= -q^{-z} \int_q^\infty \mathrm{d}p\, p^{z-1}\Big(\sum_m c_m p^m\Big) - q^{-z} \int_q^\infty \mathrm{d}p\, p^{z-1} I_R(p) \\ &= -\sum c_m q^m/(m+z) - q^{-z} \int_q^\infty \mathrm{d}p\, p^{z-1} I_R(p) = U_0(q) - s(z, I)q^{-z} \end{aligned} \tag{3.23}$$

where s is given by

$$s(z, I) = \int_0^\infty \mathrm{d}p\, p^{z-1} I_R(p). \tag{3.24}$$

For the coefficients $c_{ijk}(q)$ of equation (3.10) we require a regular behaviour at $q = 0$. Therefore we expand in accordance with equation (3.19):

$$a'_{ijk}(p, 0) = a_{ijk}^{(0)} + a_{ijk}^{(1)} p + \ldots + a_{ijk}^{(v)} p^v + a'^{R}_{ijk}(p, 0) \tag{3.25}$$

with

$$v < y_i - y_j - y_k < v + 1. \tag{3.26}$$

Then we obtain

$$c_{ijk}(\boldsymbol{q}) = \sum_{m=0}^{v} a_{ijk}^{(m)} q^m/(m - y_i + y_j + y_k) - q^{y_i - y_j - y_k} \int_0^q a'^{R}_{ijk}(\boldsymbol{p}, 0) p^{-1-y_i+y_j+y_k}\, \mathrm{d}p \tag{3.27}$$

where $\boldsymbol{p}$ points in the direction of $\boldsymbol{q}$ (the coefficients $a_{ijk}^{(m)}$ may depend on the direction of $\boldsymbol{q}$ too). Substituting $c_{ijk}(q)$ in equation (3.9) we may iterate higher-order terms in g and thus obtain the formal expansion for the scaling field f in powers of g to linear order in λ.

4. The RG equation for the perturbations $O_i(q)$

The scaling fields g_i and $f_i(q)$ in equations (3.4) and (3.6) have simple transformation properties

$$g_i(l) = e^{y_i l} g_i(0) \tag{4.1}$$

$$f_i(q, l) = e^{y_i l} f_i(q\, e^{-l}, 0). \tag{4.2}$$

Therefore the Hamiltonian

$$H_0 = H\{g_i, f_i(q)\} \tag{4.3}$$

transforms under the renormalization group into

$$H_l = H\{e^{y_i l} g_i(0), e^{y_i l} f_i(q\, e^{-l})\}. \tag{4.4}$$

Therefore the Hamiltonian

$$H_0 = H\{g\} + \sum \kappa_i O_i(q_i, \{g\}), \tag{4.5}$$

where

$$O_i(q) = \left.\frac{\partial H\{g, f\}}{\partial f_i(q)}\right|_{f=0}. \tag{4.6}$$

will transform into

$$H_l = H\{g_i\, e^{y_i l}\} + \sum \kappa_i\, e^{y_i l} O_i(q_i\, e^l, \{g\, e^{yl}\}) + O(\kappa^2). \tag{4.7}$$

We will now consider the contributions nonlinear in κ. Let us start from the Hamiltonian (4.5) and perform an infinitesimal transformation to $l = \delta$; we obtain

$$H_\delta = H\{g\, e^{y\delta}\} + \sum_i \kappa_i (1 + y_i \delta) O_i(q_i\, e^\delta, \{g\, e^{y\delta}\})$$
$$+ \frac{\delta}{2} \sum_{i'j'k'} a'_{i'j'k'}(q_j, q_k) \lambda_{j'}(q_j) \lambda_{k'}(q_k) O^*_{i'}(q_j + q_k) + O(\delta^2) \tag{4.8}$$

where the last term comes from the nonlinear contribution in equation (2.19). We express the last term in terms of κ. From

$$\sum \kappa_i(q) O_i(q) = \sum \kappa_i(q) \left.\frac{\partial H}{\partial f_i(q)}\right|_{f=0} = \sum \kappa_i(q) \left.\frac{\partial \lambda_{i'}}{\partial f_i}\right|_{f=0} \frac{\partial H}{\partial \lambda_{i'}}$$
$$= \sum \kappa_i(q) t_{i'i}(q) O^*_{i'}(q) = \sum \lambda_{i'}(q) O^*_{i'}(q) \tag{4.9}$$

with

$$t_{i'i}(q) = \left.\frac{\partial \lambda_{i'}(q)}{\partial f_i(q)}\right|_{f=0} \tag{4.10}$$

we deduce

$$\lambda_{i'}(q) = \sum_i t_{i'i}(q) \kappa_i(q) \tag{4.11}$$

and

$$O_i(q) = \sum t_{i'i}(q) O^*_{i'}(q). \tag{4.12}$$

Similarly we obtain

$$\kappa_i(q) = \sum p_{ii'}(q)\lambda_{i'}(q) \tag{4.13}$$

and

$$O^*_{i'}(q) = \sum p_{ii'}(q)O_i(q). \tag{4.14}$$

Substituting equations (4.11), (4.14) into equation (4.8) we obtain the equation for $\mathrm{d}\kappa/\mathrm{d}l$:

$$\mathrm{d}\kappa_i(q)/\mathrm{d}l = y_i\kappa_i(q)+\tfrac{1}{2}\sum A_{ijk}(q_1, q_2)\kappa_j(q_1)\kappa_k(q_2)\delta_{q,q_1+q_2} \tag{4.15}$$

with

$$A_{ijk}(q_1, q_2) = \sum p_{ii'}(q_1+q_2)a'_{i'j'k'}(q_1, q_2)t_{j'j}(q_1)t_{k'k}(q_2). \tag{4.16}$$

The matrix t is the inverse to p:

$$\sum_j p_{ij}(q)t_{jk}(q) = \sum (\partial f_i/\partial\lambda_j)(\partial\lambda_j/\partial f_k) = \delta_{ik}. \tag{4.17}$$

Therefore we obtain an expansion in powers of g from the expansion (3.10):

$$t_{ij}(q) = \delta_{ij} - \sum c_{ijk}(q)g_k + \mathrm{O}(g^2). \tag{4.18}$$

A differential equation for t can be obtained from the expression

$$\sum_{kl} t_{ik}(q\nabla + D)(p_{kl}t_{lj}) \tag{4.19}$$

and from substitution of equations (3.9), (4.17) which yields

$$(q\nabla + D - y_i + y_j)t_{ij}(q) = \sum_{ks} a'_{iks}(q, 0)t_{kj}(q)\mu_s\{g\}. \tag{4.20}$$

In §§ 2–4 we wrote down the RG equations for the inhomogeneous perturbations $\lambda_i(q)O^*_i(q)$ and transformed them to equations for perturbations $\kappa_i(q)O_i(q)$. These new perturbations depend on the thermodynamic variables via the scaling fields g (equations (4.12), (4.18)). This dependence is smooth in g and q as long as no logarithmic corrections arise. The advantage of these new perturbations is the more simple form of the equation (4.15) for the change of the amplitudes κ under the RG transformation which in linear order is much simpler than equation (2.19). In the following sections we will derive the correlation functions for the operators $O_i(q)$ starting from equation (4.15). We expect that $A_{ijk}(q_1, q_2)$ is a smooth function of the wavevectors q and of the fields g, since we expect these properties for a'_{ijk} and the matrices p and t.

5. Expansion coefficients M of the correlation functions

Let us return to our aim, the calculation of the correlation functions. We consider the correlations

$$G_{j_1j_2\ldots j_n}(q_1, q_2, \ldots q_n) = \langle O_{j_1}(q_1)O_{j_2}(q_2)\ldots O_{j_n}(q_n)\rangle \tag{5.1}$$

with $q_1+q_2+\ldots+q_n = 0$, but demand that the momenta q_j as well as the sum of any subset of the q's do not vanish (that is we do not consider exceptional momenta). Then the function (5.1) equals its cumulant and we may express it as the nth derivative of the free energy F:

$$G = \partial^n F/\partial\kappa_{j_1}(q_1)\ldots\partial\kappa_{j_n}(q_n). \tag{5.2}$$

As outlined in the introduction we apply the RG procedure to the Hamiltonian (4.5). Then the wavevectors q will grow exponentially with l. The nonlinear terms will generate perturbations $O_s((q_j+q_k)e^l)$ from the perturbations $O_j(q_j e^l)$ and $O_k(q_k e^l)$. As soon as all the wavevectors $q\,e^l$ and the sums $(q_j+q_k+\ldots)e^l$ of the subsets are large in comparison to the 'cut-off momentum' q_0 the contributions of these wavevector-dependent perturbations are small, as can be seen from the examples given by Wilson and Kogut (1974) in appendix A for the trivial fixed point. We will assume that this property holds in general. Then all of these wavevector-dependent perturbations can be neglected and only a homogeneous perturbation $\Sigma_i\,\kappa_i(l)O_i$ survives where $O_i = \partial H/\partial g_i$.

$$H_l = H\{g\,e^{yl}\} + \sum \kappa_i(l)O_i \tag{5.3}$$

$$\kappa_i(l) = \tilde{M}_{ij_1\ldots j_n}(q_1,\ldots q_n, l)\kappa_{j_1}(q_1)\ldots\kappa_{j_n}(q_n) + \ldots . \tag{5.4}$$

Since M obeys the asymptotic behaviour $\kappa_i(l) \propto e^{y_i l}$ (it coincides with f_i in first order) we expect a finite limit:

$$\lim_{l\to\infty} e^{-y_i l}\tilde{M}_{ij_1\ldots j_n}(q_1,\ldots q_n, l) = M_{ij_1\ldots j_n}(q_1,\ldots q_n). \tag{5.5}$$

After neglecting the inhomogeneous perturbations we apply the inverse RG procedure and transform the Hamiltonian (5.3) to $l = 0$ which yields

$$\tilde{H} = H\{g\} + \sum M_{ij_1\ldots j_n}(q_1,\ldots q_n)\kappa_{j_1}(q_1)\ldots\kappa_{j_n}(q_n)O_i + O(\kappa^{n+1}); \tag{5.6}$$

thus

$$\tilde{H} = H\{\tilde{g}\} \tag{5.7}$$

with

$$\tilde{g}_i = g_i + M_{ij_1\ldots j_n}(q_1,\ldots q_n)\kappa_{j_1}(q_1)\ldots\kappa_{j_n}(q_n) + O(\kappa^{n+1}). \tag{5.8}$$

If we allow for arbitrary perturbations $\Sigma\,\kappa_j(q_j)O_j(q_j)$ then we may write

$$\tilde{g}_i = g_i + \sum_n \frac{1}{n!}\sum M_{ij_1\ldots j_n}(q_1,\ldots q_n)\kappa_{j_1}(q_1)\ldots\kappa_{j_n}(q_n) \tag{5.9}$$

where we take into account that any product of the κ's appears $n!$ times in the sum (if several perturbations $O_j(q_j)$ coincide one has to introduce a corresponding symmetry factor in equation (5.8)). Since the free energy is invariant under the total of these transformations, we obtain

$$G = \frac{\partial^n F(\hat{g})}{\partial\kappa_{j_1}(q_1)\ldots\partial\kappa_{j_n}(q_n)} = \sum_i M\frac{\partial F}{\partial g_i} = \sum_i M_{ij_1\ldots j_n}(q_1,\ldots q_n)\langle O_i\rangle. \tag{5.10}$$

We may equally well express the correlation functions as

$$\begin{aligned} G^*_{k_1\ldots k_n}(q_1,\ldots q_n) &= \langle O^*_{k_1}(q_1)\ldots O^*_{k_n}(q_n)\rangle \\ &= \sum_{ij_1\ldots j_n h} M_{ij_1\ldots j_n}(q_1,\ldots q_n)p_{j_1k_1}(q_1)\ldots p_{j_nk_n}(q_n)t_{hi}\langle O^*_h\rangle. \end{aligned} \tag{5.11}$$

To analyse the non-analytic behaviour of the correlation functions we need an equation for the coefficients M. Since both Hamiltonians (4.5) and (4.8) yield the same

Hamiltonian (5.3), we obtain

$$\tilde{M}_{ij_1\ldots j_n}(q_1,\ldots q_n,\{g\},l)$$
$$= e^{y_J\delta}\tilde{M}_{ij_1\ldots j_n}(q_1 e^{\delta}\ldots,\{g\, e^{y\delta}\},l-\delta)$$
$$+\delta\sum_s A_{sj_1j_2}(q_1,q_2)\tilde{M}_{isj_3\ldots j_n}(q_1+q_2,q_3\ldots,\{g\},l)+\ldots \tag{5.12}$$

to first order in δ with the abbreviation

$$y_J = y_{j_1}+\ldots+y_{j_n}. \tag{5.13}$$

The last term and $+\ldots$ on the right-hand side of equation (5.12) stand for the terms made up from the $\frac{1}{2}n(n-1)$ pairs $(i_1,i_2),(i_1,i_3),\ldots(i_{n-1},i_n)$. We make use of the asymptotic behaviour (5.5) of $\tilde{M}$ for large l and obtain

$$M_{ij_1\ldots j_n}(q_1,\ldots q_n,\{g\})$$
$$= e^{(y_J-y_i)\delta}M_{ij_1\ldots j_n}(q_1 e^{\delta}\ldots,\{g\, e^{y\delta}\})+\delta\sum_s A_{sj_1j_2}(q_1,q_2)M_{is}. \tag{5.14}$$

Differentiating with respect to δ yields the equation

$$(\sum q_j\nabla_j+D+y_J-y_i)M_{ij_1\ldots j_n}(q_1,\ldots q_n)$$
$$= -\sum_s A_{sj_1j_2}(q_1,q_2)M_{isj_3\ldots j_n}(q_1+q_2,q_3\ldots,\{g\})-\ldots. \tag{5.15}$$

We note that

$$M_{ij} = \delta_{ij} \tag{5.16}$$

since a homogeneous perturbation $\kappa_i O_i$ transforms into itself, and that

$$\lim_{q\to\infty} M_{ij_1\ldots j_n}(q_1,\ldots q_n) = 0 \qquad \text{for } n>1, \tag{5.17}$$

since the nonlinear terms do not contribute for $|q_j| \gg q_0$. One can easily integrate equation (5.15) with the boundary conditions (5.17) since it is of the type (3.15). With the substitution

$$p = bq \tag{5.18}$$

we find

$$M_{ij_1\ldots j_n}(q_1,\ldots q_n,g)$$
$$=\sum_s\int_1^{\infty} db\, b^{y_J-y_i-1}A_{sj_1j_2}(q_1 b,q_2 b,\{gb^y\})$$
$$\times M_{isj_3\ldots j_n}((q_1+q_2)b,q_3 b,\ldots\{gb^y\})+\ldots. \tag{5.19}$$

From this recurrence relation and the initial condition (5.16) we may iterate the expansion coefficients M to arbitrary order n.

From equations (4.15), (5.2) and $F(H_0) = e^{-dl}F(H_l)$ one derives similarly the equation for the correlation function:

$$(D+\sum q_j\nabla_j+y_J-d)G_{j_1\ldots j_n}(q_1,\ldots q_n)$$
$$= -\sum A_{sj_1j_2}(q_1,q_2)G_{sj_3\ldots j_n}(q_1+q_2,q_3\ldots q_n)-\text{permutations}. \tag{5.20}$$

This equation differs from equation (5.15) in so far as it can be applied for any momenta whereas equation (5.15) applies only for non-exceptional momenta. In applying equation (5.20) we replace continuously products of operators by single operators. This corresponds to the idea of the operator product expansion and the reduction hypothesis (Wilson 1969, Kadanoff 1969, Polyakov 1969).

6. Two-point correlation functions

We apply the results of § 5 to two-point correlation functions. Equation (5.15) yields with condition (5.16) to the equation

$$(q\partial/\partial q + D + y_j + y_m - y_i)M_{ijm}(q, -q) = -A_{ijm}(q, -q) \tag{6.1}$$

with the solution

$$M_{ijm}(q, -q, \{g\}) = q^{y_i - y_j - y_m} \int_q^\infty \mathrm{d}p \, p^{y_j + y_m - y_i - 1} A_{ijm}(p, -p, g(p/q)^y). \tag{6.2}$$

Let us expand A in powers of g:

$$A_{ijm}(p, -p, g) = \sum_{\mathrm{K}} c_{ijm\mathrm{K}}(p) g_{\mathrm{K}} \tag{6.3}$$

where g_{K} stands for any product of g's,

$$g_{\mathrm{K}} = g_{k_1} g_{k_2} \cdots g_{k_r}. \tag{6.4}$$

Then we obtain

$$M_{ijm} = \sum_{\mathrm{K}} q^{y_i - y_j - y_m - y_{\mathrm{K}}} g_{\mathrm{K}} \int_q^\infty \mathrm{d}p \, p^{-y_i + y_j + y_m + y_{\mathrm{K}} - 1} c_{ijm\mathrm{K}}(p). \tag{6.5}$$

The functions

$$q^{-z} \int_q^\infty \mathrm{d}p \, p^{z-1} c(p)$$

with

$$z = -y_i + y_j + y_m + y_{\mathrm{K}} \tag{6.6}$$

are the solutions U_∞ of the differential equation (3.15):

$$(q\,\partial/\partial q + z)U(q) = c(q). \tag{6.7}$$

Since we are interested in the behaviour of $M(q)$ for small q we use equation (3.23):

$$U_\infty(q) = U_0(q) - sq^{-z} \tag{6.8}$$

and obtain

$$M_{ijm} = R_{ijm} + S_{ijm}, \tag{6.9}$$

where R is the solution of equation (6.1) which is regular at $q = 0$:

$$R_{ijm}(q, \{g\}) = \sum_{\mathrm{K}} r_{ijm\mathrm{K}}(q) g_{\mathrm{K}} \tag{6.10}$$

and S is the solution

$$S_{ijm}(q, \{g\}) = q^{y_i - y_j - y_m} S_{ijm}(e, \{g/q^y\}) = q^{y_i - y_j - y_m} \sum_{\mathrm{K}} s_{ijm\mathrm{K}} g_{\mathrm{K}}/q^{y_{\mathrm{K}}} \tag{6.11}$$

of the homogeneous equation with

$$s_{ijmK} = \int_0^\infty \mathrm{d}p\, p^{y_j + y_m + y_K - y_i - 1} c^R_{ijmK}(p). \tag{6.12}$$

In general s_{ijmK} depends on the direction $e = q/|q|$. Again we do not discuss any logarithmic factors which may arise if $y_i - y_j - y_m - y_K$ is a non-negative integer.

Therefore the two-point correlation function can be written:

$$G_{jm}(q, -q) = \langle O_j(q) O_m(-q) \rangle = \sum_i (R_{ijm}(q, \{g\}) + q^{y_i - y_j - y_m} S_{ijm}(e, \{g/q^y\})) \langle O_i \rangle \tag{6.13}$$

where R and S are completely determined from the functions a'_{ijk} in equation (2.16). The second contribution to G,

$$G^s_{jm}(q, -q) = \sum_i q^{y_i - y_j - y_m} S_{ijm}(e, \{g/q^y\}) \langle O_i \rangle, \tag{6.14}$$

obeys scaling. If we multiply q by b, g_i by b^{y_i}, then G^s is multiplied by $b^{d-y_j-y_m}$ since $\langle O_i \rangle = \partial F/\partial g_i$ scales like b^{d-y_i}, which follows from

$$F\{g_i b^{y_i}\} = b^d F\{g_i\} \tag{6.15}$$

(note that O_i according to the definition $O_i = \partial H/\partial g_i$ depends on the variables g_i). If we approach the critical point then the scaling fields of the relevant operators tend to zero ($g_i^{\mathrm{rel}} \to 0$), whereas the scaling fields g_i^{irr} of the irrelevant operators approach some constant. Therefore normally one considers the scaling behaviour for the transformation $q \to bq$, $g_i^{\mathrm{rel}} \to b^{y_i} g_i^{\mathrm{rel}}$, $g_i^{\mathrm{irr}} \to g_i^{\mathrm{irr}}$. Since the scaling exponent y_i is positive for relevant and negative for irrelevant operators, we find in the limit $b \to 0$ that the contributions from g_i^{rel} in the scaling function survive whereas the contributions from g^{irr} yield corrections to scaling. The leading scaling behaviour is given by

$$G^{\mathrm{scal}}_{jm}(q, -q) = \sum_i q^{y_i - y_j - y_m} S_{ijm}(e, \{g^{\mathrm{rel}}/q^y\}) \langle O_i \rangle_{\mathrm{scal}}. \tag{6.16}$$

All operators O_i contribute to the scaling behaviour. However, it is only necessary to consider these expectation values as functions of the relevant scaling fields g_i^{rel}, since the irrelevant fields yield (under certain restrictions, see Wegner 1972) only corrections to the leading scaling behaviour. If $y_j + y_m > d$ then the scaling part of the correlation function (6.16) dominates the 'regular' contribution:

$$G^{\mathrm{reg}}_{jm}(q, -q) = \sum_i R_{ijm}(q, \{g\}) \langle O_i \rangle. \tag{6.17}$$

If however $y_j + y_m < d$ then the regular contribution may yield the leading term. If for example the critical exponent α of the specific heat is negative ($2 - \alpha = d/y_{\mathrm{E}}$), then the energy–energy correlation does not become singular at the critical point since it scales like $q^{d-2y_{\mathrm{E}}} = q^{-\alpha y_{\mathrm{E}}}$.

Equation (6.16) allows an expansion of the correlation function in (fractional) powers of q^{-1} in the critical region. In a system with two relevant operators O_{E} and O_h (crudely speaking energy and magnetization) one obtains contributions

$$\left(\frac{g_{\mathrm{E}}}{q^{y_{\mathrm{E}}}}\right)^{n_{\mathrm{E}}} \left(\frac{g_h}{q^{y_h}}\right)^{n_h} q^{y_i - y_j - y_m} \langle O_i \rangle \tag{6.18}$$

where in linear order the scaling field g_{E} is the temperature difference from the critical temperature, and g_h is the magnetic field (not all of the terms (6.18) will appear because of symmetries: symmetry of the system under reversal or rotation of the order parameter, conformal invariance).

For the spin–spin correlation function one obtains among others the contributions

$$q^{-2+\eta} \qquad \text{for } n_{\mathrm{E}} = n_h = 0, \qquad O_i = 1 \tag{6.19a}$$

with $2-\eta = 2y_h - d$,

$$q^{-2+\eta-1/\nu}\tau \qquad \text{for } n_{\mathrm{E}} = 1, \qquad n_h = 0, \qquad O_i = 1 \tag{6.19b}$$

$$q^{-2+\eta-(1-\alpha)/\nu}\langle O_{\mathrm{E}}\rangle \qquad \text{for } n_{\mathrm{E}} = n_h = 0 \tag{6.19c}$$

as proposed by Fisher and Langer (1968). Fisher and Aharony (1973) showed that these terms are consistent with an expansion of the correlation function to second order in $\epsilon = 4-d$. These terms were derived in an ϵ expansion from the Callan–Symanzik equation by Brezin *et al* (1974a). Besides, Brezin *et al* (1974b) derived a contribution proportional to

$$q^{-2-\eta-d+\phi/\nu}\langle S_\alpha S_\beta - \delta_{\alpha\beta}S^2/n\rangle \tag{6.19d}$$

for the n-vector model ($n > 1$), where ϕ is the cross-over exponent for anisotropic perturbations (Riedel and Wegner 1969). Hecht (1967) calculated the spin–energy correlation function of the two-dimensional Ising model in zero magnetic field. He obtained in leading order in q^{-1} the term

$$\langle S\rangle q^{-1} \tag{6.20}$$

which agrees with equation (6.18) for $n_{\mathrm{E}} = n_h = 0$, since $y_{\mathrm{E}} = 1$ in this model.

7. Scaling fields, three-point correlations

In §§ 3, 4 we calculated the scaling fields f to first order in λ. We called the operators

$$\left.\frac{\partial H\{g, f\}}{\partial f_i(q)}\right|_{f=0} = O_i(q) \tag{4.6}$$

and expressed the perturbations to the translational invariant Hamiltonian $H\{g\}$ in terms of the fields κ:

$$H = H\{g\} + \sum_i \kappa_i(q_i) O_i(q_i). \tag{4.5}$$

In this section we expand the scaling fields f in powers of κ:

$$\begin{aligned} f_i(q) = {} & \kappa_i(q) + \tfrac{1}{2}\sum P_{ijk}(q_1, q_2)\kappa_j(q_1)\kappa_k(q_2)\delta_{q,q_1+q_2} \\ & + \frac{1}{3!}\sum P_{ijkm}(q_1, q_2, q_3)\kappa_j(q_1)\kappa_k(q_2)\kappa_m(q_3)\delta_{q,q_1+q_2+q_3} + \mathrm{O}(\kappa^4). \end{aligned} \tag{7.1}$$

To do this we calculate the derivative of equation (7.1) with respect to l and insert the equations of motion (3.6), (4.15):

$$\begin{aligned} & y_i\kappa_i(q) + \tfrac{1}{2}y_i \sum P_{ijk}(q_1, q_2)\kappa_j(q_1)\kappa_k(q_2)\delta_{q,q_1+q_2} \\ & \quad = y_i\kappa_i(q) + \tfrac{1}{2}\sum A_{ijk}(q_1, q_2)\kappa_j(q_1)\kappa_k(q_2)\delta_{q,q_1+q_2} \\ & \qquad + \tfrac{1}{2}\sum (y_j + y_k + q_1\nabla_1 + q_2\nabla_2 + D)P_{ijk}\kappa_j(q_1)\kappa_k(q_2) + \mathrm{O}(\kappa^3) \end{aligned} \tag{7.2}$$

which yields the differential equation

$$(q_1\nabla_1 + q_2\nabla_2 + D + y_j + y_k - y_i)P_{ijk}(q_1, q_2, \{g\}) = -A_{ijk}(q_1, q_2). \tag{7.3}$$

We demand that the scaling fields show a regular behaviour at $q_1 = q_2 = 0$. The solution can be found similar to that of equation (6.1). For $q_1 = -q_2 = q$ equation (7.3) reduces to

$$(q\nabla + D + y_j + y_k - y_i)P_{ijk}(q, -q) = -A_{ijk}(q, -q) \tag{7.4}$$

which is the equation for M_{ijk}. Since however $P_{ijk}(q, -q, \{g\})$ has to be regular at $q = 0$ we have from equation (6.9) the solution

$$P_{ijk}(q, -q) = R_{ijk}(q). \tag{7.5}$$

Collecting the terms of order κ^3 in equation (7.2) we obtain

$$Q_3 P_{ijkm}(q_1, q_2, q_3) = -\sum P_{ijs}(q_1, q_2 + q_3)A_{skm}(q_2, q_3) - \text{permutations} \tag{7.6}$$

with

$$Q_3 = q_1\nabla_1 + q_2\nabla_2 + q_3\nabla_3 + D + y_j + y_k + y_m - y_i. \tag{7.7}$$

Again we choose the solution P regular at $q_1 = q_2 = q_3 = 0$.

Now we discuss the derivatives of the free energy F with respect to the scaling fields f. The free energy obeys for a fixed volume (in the thermodynamic limit) the relation

$$F(H(l)) = e^{dl} F(H(0)). \tag{7.8}$$

Since f_i and g_i transform according to equations (3.4) and (3.6) we obtain for F in terms of f and g:

$$F\{g\,e^{yl}, f(g\,e^{l}, l) = e^{yl}f(g, 0)\} = e^{dl} F\{g, f(q, 0)\}. \tag{7.9}$$

Differentiating equation (7.9) with respect to $f_i(q)$ yields the scaling law

$$\partial^n F\{g\,e^{yl}\}/\partial f_i(q_1\,e^{l})\ldots = e^{(d - y_i - \ldots)l}\,\partial^n F/\partial f_i(q_1)\ldots. \tag{7.10}$$

Therefore we obtain

$$\begin{aligned}\langle O_i(q)O_j(-q)\rangle &= \frac{\partial^2 F}{\partial\kappa_i(q)\partial\kappa_j(-q)} \\ &= \sum \frac{\partial^2 f_k}{\partial\kappa_i(q)\partial\kappa_j(-q)}\frac{\partial F}{\partial f_k} + \sum \frac{\partial f_{i'}}{\partial\kappa_i}\frac{\partial f_{j'}}{\partial\kappa_j}\frac{\partial^2 F}{\partial f_{i'}\partial f_{j'}} \\ &= \sum_k P_{kij}(q, -q)\langle O_k\rangle + \sum \frac{\partial^2 F}{\partial f_i(q)\partial f_j(-q)},\end{aligned} \tag{7.11}$$

where we used equation (7.1) and $\partial F/\partial f_k = \langle O_k\rangle$. Comparison with equations (6.9), (7.5) shows that

$$\partial^2 F/\partial f_i(q)\partial f_j(-q) = \sum_k S_{kij}(q, \{g\})\langle O_k\rangle \tag{7.12}$$

which indeed is a homogeneous function. Ma (1974) has pointed out that starting from an equation like (7.9) one finds that $O_i(q)O_j(-q)$ does not show the exact scaling behaviour but $O_i(q)O_j(-q) + O_{ij}(q, -q)$ does, where

$$O_{ij}(q, -q) = \partial^2 H/\partial f_i(q)\partial f_j(-q). \tag{7.13}$$

From equation (7.1) one immediately finds that

$$\kappa_i(q) = f_i(q) - \tfrac{1}{2} \sum P_{ijk}(q_1, q_2) f_j(q_1) f_k(q_2) \delta_{q, q_1 + q_2} \tag{7.14}$$

which yields

$$O_{ij}(q, -q) = -\sum_k P_{kij}(q, -q) O_k, \tag{7.15}$$

which confirms Ma's result.

Let us go one step further and shortly consider the three-point correlation functions. From equation (5.15) we obtain $(q_1 + q_2 + q_3 = 0)$:

$$Q_3 M_{ijkm}(q_1, q_2, q_3) = -\sum_s M_{ijs}(q_1, -q_1) A_{skm}(q_2, q_3) - \text{permutations.} \tag{7.16}$$

One obtains

$$\begin{aligned} M_{ijkm}(q_1, q_2, q_3) &= R_{ijkm}(q_1, q_2, q_3) + S_{ijkm}(q_1, q_2, q_3) \\ &\quad + \sum S_{ijs}(q_1) P_{skm}(q_2, q_3) + \text{permutations.} \end{aligned} \tag{7.17}$$

One obtains the last terms on the right-hand side of equation (7.17) from the contributions of S_{ijs} to M_{ijs} where we make use of the homogeneity of the function S. R_{ijkm} is the solution, regular at $q_1 = q_2 = q_3 = 0$, of

$$Q_3 R_{ijkm} = -\sum R_{ijs}(q_1) A_{skm}(q_2, q_3) - \text{permutations} \tag{7.18}$$

and S_{ijkm} obeys the homogeneous equation

$$Q_3 S_{ijkm} = 0. \tag{7.19}$$

It is defined uniquely by the boundary condition (5.17). On the other hand one shows by evaluating the terms of order κ^3 in equation (7.2) and comparing with equation (7.18) that

$$P_{ijkm} = R_{ijkm}. \tag{7.20}$$

One can now easily verify that

$$\partial^3 F / \partial f_j(q_1) \partial f_k(q_2) \partial f_m(q_3) = \sum_i S_{ijkm}(q_1, q_2, q_3) \langle O_i \rangle. \tag{7.21}$$

We may finally express g_i of equation (1.2) in terms of the scaling fields f_i. Using equations (6.9), (7.1), (7.18) and noting that equation (7.1) applies also for $q = 0$ with vanishing $\kappa_i(q = 0)$, we find that

$$\tilde{g}_i = g_i + f_i + \tfrac{1}{2} \sum S_{ijk}(q, -q) f_j(q) f_k(-q) + \dots . \tag{7.22}$$

The general picture which emerges from this calculation is the following: the free energy of the Hamiltonian (1.1) with inhomogeneous perturbations is equal to the free energy of a translational invariant Hamiltonian $H\{\tilde{g}\}$ where $\tilde{g}$ is given by the expansion (1.2). This allows the calculation of wavevector-dependent correlations in terms of expectation values of homogeneous operators. The expansion coefficients M of g obey differential equations (5.15) which can be calculated from the expansion coefficients A of the renormalization group equation (4.15). The coefficients M can be expressed in terms of a function R which is regular for small q and g, and a singular homogeneous function S (equations (6.9), (7.19)). If one introduces the scaling fields f then the functions R and S assume a meaning of their own (although they are related by the boundary condition (5.17)). The functions R are now the expansion coefficients of f in powers of the fields κ (equation (7.1)), and the functions S are the expansion coefficients of $\mathfrak{g}_i$ in powers of the scaling fields f_i (equation (7.22)).

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